Parallelism Structure on a Smooth Manifold

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Abstract

Using the theory of extensors developed in a previous paper we present a theory of the parallelism structure on arbitrary smooth manifold. Two kinds of Cartan connection operators are introduced and both appear in intrinsic versions (i.e., frame independent) of the first and second Cartan structure equations. Also, the concept of deformed parallelism structures and relative parallelism structures which play important role in the understanding of geometrical theories of the gravitational field are investigated.

1 Introduction

In this article using mainly the algebraic tools developed in [7, 6] we present a theory of a general parallelism structure on an arbitrary real differentiable manifold M of dimension n. Section 2 recalls briefly the concepts of covariant derivatives of vector, form and extensor fields. Section 3 introduce *two* kinds of Cartan connection operators, namely the *plus* and *minus* Cartan connections. With these conceptions we show in Section 4 an intrinsic version of Cartan first structure equation for a biform valued (1,2) extensor field Θ (torsion) which involves the minus connection. In section 5 we show an intrinsic Cartan second structure equation for a (1 vector and 1 form) biform valued curvature extensor Ω where both the plus and minus connection naturally appears! In Section 7 we study the concept of a symmetric parallelism structure and in Section 8 we present the important concept of deformed parallelism structures. In Section 8 we introduce the concept of relative parallelism structure. These concepts play an important role in any deep study of geometric theories of the gravitational field. In Section 10 we present our conclusions. The present paper and the sequel ones [8, 9] constitutes a valuable and simplifying improvement over the presentation given in [2, 3, 4, 5] which uses only the geometric algebra of multivector fields and $\bigwedge TM$ valued extensor fields.

2 Parallelism Structure

Let U be an open set of the smooth manifold M (i.e., $U \subseteq M$). The set of smooth[1] scalar fields on U, as well-known, has a natural structure of *ring* (*with identity*), and it will be denoted by $\mathcal{S}(U)$. The set of smooth vector fields on U and the set of smooth form fields on U, as well-known, have natural structure of *modules over* $\mathcal{S}(U)$.

A smooth 2 vector variables vector operator field on U,

$$\Gamma : \mathcal{V}(U) \times \mathcal{V}(U) \longrightarrow \mathcal{V}(U),$$

such that it satisfies the following axioms:

[1]Smooth in this paper means i.e., $\mathcal{C}^{\infty}$-differentiable or at least enough differentiable in order for our statements to hold.

i. for all $f, g \in \mathcal{S}(U)$ and $a, b, v \in \mathcal{V}(U)$

$$\Gamma(fa + gb, v) = f\Gamma(a, v) + g\Gamma(b, v), \tag{1}$$

ii. for all $f, g \in \mathcal{S}(U)$ and $a, v, w \in \mathcal{V}(U)$

$$\Gamma(a, fv + gw) = (af)v + (ag)w + f\Gamma(a, v) + g\Gamma(a, w), \tag{2}$$

is called a *connection* on U.

The behavior of Γ with respect to the first variable will be called *strong linearity*, and the behavior of Γ with respect to the second variable will be called *quasi linearity*.

The algebraic pair $\langle U, \Gamma \rangle$, where U is an open set of the smooth manifold M and Γ is a connection on U, will be called a *parallelism structure* on U.

2.1 Covariant Derivative of Vector and Form Fields

Let $\langle U, \Gamma \rangle$ be a parallelism structure on U. Let us take $a \in \mathcal{V}(U)$. The *a-Directional Covariant Derivative* (*a-DCD*) of a smooth vector field on U, associated with $\langle U, \Gamma \rangle$, is the mapping

$$\mathcal{V}(U) \ni v \longmapsto \nabla_a v \in \mathcal{V}(U)$$

such that

$$\nabla_a v = \Gamma(a, v). \tag{3}$$

The *a-DCD* of a smooth form field on U, is the mapping

$$V^*(U) \ni \omega \longmapsto \nabla_a \omega \in V^*(U)$$

such that for every $v \in \mathcal{V}(U)$

$$\nabla_a \omega(v) = a\omega(v) - \omega(\nabla_a v). \tag{4}$$

Eq.(1) and Eq.(2), according with Eq.(3), imply two basic properties for the covariant derivative of smooth vector fields:

- For all $f \in S(U)$, and $a, b, v \in \mathcal{V}(U)$

$$\begin{aligned} \nabla_{a+b} v &= \nabla_a v + \nabla_b v, \\ \nabla_{fa} v &= f \nabla_a v. \end{aligned} \tag{5}$$

- For all $f \in S(U)$, and $a, v, w \in \mathcal{V}(U)$

$$\begin{aligned} \nabla_a(v + w) &= \nabla_a v + \nabla_a w, \\ \nabla_a(fv) &= (af)v + f\nabla_a v. \end{aligned} \tag{6}$$

From Eq.(4), using the strong linearity of the smooth form fields and recalling the Leibniz property of the a-directional derivative of the smooth scalar fields, it follows two basic properties for the covariant derivative of the smooth form fields:

- For all $f \in S(U)$, and $a, b \in \mathcal{V}(U)$, and $\omega \in \mathcal{V}^*(U)$

$$\begin{aligned} \nabla_{a+b}\omega &= \nabla_a\omega + \nabla_b\omega, \\ \nabla_{fa}\omega &= f\nabla_a\omega. \end{aligned} \tag{7}$$

- For all $f \in S(U)$, and $a \in \mathcal{V}(U)$ and $\omega, \sigma \in \mathcal{V}^*(U)$

$$\begin{aligned} \nabla_a(\omega + \sigma) &= \nabla_a\omega + \nabla_a\sigma, \\ \nabla_a(f\omega) &= (af)\omega + f\nabla_a\omega. \end{aligned} \tag{8}$$

2.2 Covariant Derivative of Elementary Extensor Fields

We recall that there exist exactly two types of *elementary* extensors over a real vector space V of finite dimension[2], i.e., the linear mappings of the type

$$\begin{aligned} &\underbrace{V \times \cdots \times V}_{k\text{-copies}} \times \underbrace{V^* \times \cdots \times V^*}_{l\text{-copies}} \ni (v_1, \ldots, v_k, \omega^1, \ldots, \omega^l) \\ &\qquad\qquad \longmapsto \tau(v_1, \ldots, v_k, \omega^1, \ldots, \omega^l) \in V \end{aligned} \tag{9}$$

called a *k-covariant and l-contravariant vector extensor over V*, and the linear mappings of the type

$$\begin{aligned} &\underbrace{V \times \cdots \times V}_{k\text{-copies}} \times \underbrace{V^* \times \cdots \times V^*}_{l\text{-copies}} \ni (v_1, \ldots, v_k, \omega^1, \ldots, \omega^l) \\ &\qquad\qquad \longmapsto \tau(v_1, \ldots, v_k, \omega^1, \ldots, \omega^l) \in V^* \end{aligned} \tag{10}$$

[2] V could be also a module over a ring.

called a *k-covariant and l-contravariant form extensor over V*.

The set of each of them are denoted by $ext_k^l(V)$ and $\overset{*}{ext}{}_k^l(V)$, and have natural structures of real vector spaces and moreover

$$\dim ext_k^l(V) = \dim \overset{*}{ext}{}_k^l(V) = n^{k+l+1}. \tag{11}$$

Let U be an open set of the manifold M (i.e., $U \subseteq M$). A mapping

$$U \ni p \longmapsto \tau_{(p)} \in ext_k^l(T_pM) \tag{12}$$

is called a *k-covariant and l-contravariant vector extensor field* on U.

Such a extensor field τ is said to be *smooth* if and only if for all $v_1, \ldots, v_k \in \mathcal{V}(U)$, and $\omega^1, \ldots, \omega^l \in \mathcal{V}^*(U)$ the mapping

$$U \ni p \longmapsto \tau_{(p)}(v_{1(p)}, \ldots, v_{k(p)}, \omega^1_{(p)}, \ldots, \omega^l_{(p)}) \in T_pM \tag{13}$$

is a smooth vector field on U.

A mapping

$$U \ni p \longmapsto \tau_{(p)} \in \overset{*}{ext}{}_k^l(T_pM) \tag{14}$$

is called a *k-covariant and l-contravariant form extensor field* on U.

Such a extensor field τ is said to be *smooth* if and only if for all $v_1, \ldots, v_k \in \mathcal{V}(U)$, and $\omega^1, \ldots, \omega^l \in \mathcal{V}^*(U)$ the mapping

$$U \ni p \longmapsto \tau_{(p)}(v_{1(p)}, \ldots, v_{k(p)}, \omega^1_{(p)}, \ldots, \omega^l_{(p)}) \in T_p^*M \tag{15}$$

is a smooth form field on U.

The sets of these smooth extensor field on U are denoted by $ext_k^l\mathcal{V}(U)$ and $\overset{*}{ext}{}_k^l\mathcal{V}(U)$, since according with the definitions of smoothness for extensor fields on U as given above, each of smooth extensor field on U could be properly seen as some type of *extensor over* $\mathcal{V}(U)$.

Let $\langle U, \Gamma \rangle$ be a parallelism structure on U. Take $a \in \mathcal{V}(U)$.

The *a-DCD of a smooth k-covariant and l-contravariant vector* (*or form*) *extensor field* on U, and associated with $\langle U, \Gamma \rangle$, are defined by

$$ext_k^l\mathcal{V}(U) \ni \tau \longmapsto \nabla_a\tau \in ext_k^l\mathcal{V}(U) \text{ and } \overset{*}{ext}{}_k^l\mathcal{V}(U) \ni \tau \longmapsto \nabla_a\tau \in \overset{*}{ext}{}_k^l\mathcal{V}(U),$$

respectively, such that for every $v_1, \ldots, v_k \in \mathcal{V}(U)$, and $\omega^1, \ldots, \omega^l \in V^*(U)$:

$$\begin{aligned}\nabla_a \tau(v_1, \ldots, v_k, \omega^1, \ldots, \omega^l) = {} & \nabla_a(\tau(v_1, \ldots, v_k, \omega^1, \ldots, \omega^l)) \\ & - \tau(\nabla_a v_1, \ldots, v_k, \omega^1, \ldots, \omega^l) - \cdots \\ & - \tau(v_1, \ldots, \nabla_a v_k, \omega^1, \ldots, \omega^l) \\ & - \tau(v_1, \ldots, v_k, \nabla_a \omega^1, \ldots, \omega^l) - \cdots \\ & - \tau(v_1, \ldots, v_k, \omega^1, \ldots, \nabla_a \omega^l). \end{aligned} \tag{16}$$

We present two basic properties for the covariant derivative of smooth extensor fields:

- For $f \in \mathcal{S}(U)$, and $a, b \in \mathcal{V}(U)$, and $\tau \in ext_k^l \mathcal{V}(U)$ (or $\tau \in \overset{*}{ext}{}_k^l \mathcal{V}(U)$)

$$\begin{aligned}\nabla_{a+b}\tau &= \nabla_a \tau + \nabla_b \tau \\ \nabla_{fa}\tau &= f \nabla_a \tau. \end{aligned} \tag{17}$$

- For $f \in \mathcal{S}(U)$, and $a \in \mathcal{V}(U)$, and $\tau, \sigma \in ext_k^l \mathcal{V}(U)$ (or $\tau, \sigma \in \overset{*}{ext}{}_k^l \mathcal{V}(U)$)

$$\begin{aligned}\nabla_a(\tau + \sigma) &= \nabla_a \tau + \nabla_a \sigma, \\ \nabla_a(f\tau) &= (af)\tau + f \nabla_a \tau. \end{aligned} \tag{18}$$

3 Cartan Connections

The smooth 1 *vector and* 1 *form variables form operator field* on U,

$$\Gamma^+ : \mathcal{V}(U) \times \mathcal{V}^*(U) \longrightarrow V^*(U)$$

such that

$$\Gamma^+(v, \omega) = \langle \omega, \nabla_{e_\sigma} v \rangle \, \varepsilon^\sigma, \tag{19}$$

where $\{e_\mu, \varepsilon^\mu\}$ is any pair of dual frame fields on $V \supseteq U$, will be called *plus Cartan connection* on U. We emphasis that Γ^+, as a mapping associated with Γ is well-defined, since the form field $\Gamma^+(v, \omega)$ does not depend on the pair $\{e_\mu, \varepsilon^\mu\}$ which is chosen for calculating it.

The plus Cartan connection has the basic properties:

- For all $f \in \mathcal{S}(U)$ and $v, w \in \mathcal{V}(U)$ and $\omega \in \mathcal{V}^*(U)$

$$\begin{aligned} \Gamma^+(v + w, \omega) &= \Gamma^+(v, \omega) + \Gamma^+(w, \omega), \\ \Gamma^+(fv, \omega) &= \langle \omega, v \rangle \, df + f\Gamma^+(v, \omega). \end{aligned} \tag{20}$$

  We note that Γ^+ has *quasi linearity* with respect to the first variable.

- For all $f \in S(U)$ and $\omega, \sigma \in \mathcal{V}^*(U)$ and $v \in \mathcal{V}(U)$

$$\begin{aligned} \Gamma^+(v, \omega + \sigma) &= \Gamma^+(v, \omega) + \Gamma^+(v, \sigma), \\ \Gamma^+(v, f\omega) &= f\Gamma^+(v, \omega). \end{aligned} \tag{21}$$

  We note that Γ^+ has *strong linearity* with respect to the second variable.

- Γ^+ has a kind of *inversion property*

$$\left\langle \Gamma^+(v, \omega), a \right\rangle = \langle \omega, \nabla_a v \rangle \, . \tag{22}$$

Then,

$$\nabla_a v = \left\langle \Gamma^+(v, \varepsilon^\sigma), a \right\rangle e_\sigma, \tag{23}$$

which gives $\nabla_a v$ (the *a-DCD* of a smooth vector field v) in terms of $\Gamma^+(v, \omega)$. It should be noted that the strong linearity of Γ^+ with respect to the second variable is essential in the realization of the *frame field independent character* on the right side of Eq.(23), what is logically necessary for consistency.

The smooth 1 *vector and* 1 *form variables form operator field* on U,

$$\Gamma^- : \mathcal{V}(U) \times \mathcal{V}^*(U) \longrightarrow V^*(U)$$

such that

$$\Gamma^-(v, \omega) = \langle \nabla_{e_\sigma} \omega, v \rangle \, \varepsilon^\sigma, \tag{24}$$

where as before $\{e_\mu, \varepsilon^\mu\}$ is any pair of dual frame field on $V \supseteq U$, will be called *minus Cartan connection* on U. We emphasis that Γ^-, as mapping associated with Γ, is also well-defined.

The minus Cartan connection has the basic properties:

- For all $f \in \mathcal{S}(U)$ and $v, w \in \mathcal{V}(U)$ and $\omega \in \mathcal{V}^*(U)$

$$\begin{aligned} \Gamma^-(v + w, \omega) &= \Gamma^-(v, \omega) + \Gamma^-(w, \omega), \\ \Gamma^-(fv, \omega) &= f\Gamma^-(v, \omega). \end{aligned} \tag{25}$$

  That is, Γ^- has *strong linearity* with respect to the first variable.

- For all $f \in S(U)$ and $\omega, \sigma \in \mathcal{V}^*(U)$ and $v \in \mathcal{V}(U)$

$$\begin{aligned} \Gamma^-(v, \omega + \sigma) &= \Gamma^-(v, \omega) + \Gamma^-(v, \sigma), \\ \Gamma^-(v, f\omega) &= \langle \omega, v \rangle \, df + f\Gamma^-(v, \omega). \end{aligned} \tag{26}$$

  That is, Γ^- has *quasi linearity* with respect to the second variable.

- Γ^- has also a kind of *inversion property*, i.e.,

$$\left\langle \Gamma^-(v, \omega), a \right\rangle = \langle \nabla_a \omega, v \rangle \, . \tag{27}$$

Then,

$$\nabla_a \omega = \left\langle \Gamma^-(e_\sigma, \omega), a \right\rangle \varepsilon^\sigma. \tag{28}$$

which gives $\nabla_a \omega$ (the *a-DCD* of a smooth form field ω) in terms of $\Gamma^-(v, \omega)$. We emphasize that the strong linearity of Γ^- with respect to the first variable is essential for logical consistency of Eq.(28).

The relationship between Γ^+ and Γ^- is given by the noticeable property

- $$\Gamma^+(v, \omega) + \Gamma^-(v, \omega) = d \, \langle \omega, v \rangle \, . \tag{29}$$

The well-known *Cartan connections forms* associated with some pair of dual frame fields $\{e_\mu, \varepsilon^\mu\}$, here denoted by γ_μ^ν, are just the values of Γ^+ and Γ^- evaluated for (e_μ, ε^μ), i.e.,

$$\gamma_\mu^\nu = \Gamma^+(e_\mu, \varepsilon^\nu) = -\Gamma^-(e_\mu, \varepsilon^\nu). \tag{30}$$

4 Torsion

Let $\langle U, \Gamma \rangle$ be a parallelism structure on U. The smooth 2-*covariant vector extensor field* on U, defined by

$$\mathcal{V}(U) \times \mathcal{V}(U) \ni (a, b) \longmapsto \tau(a, b) \in \mathcal{V}(U)$$

such that

$$\tau(a, b) = \nabla_a b - \nabla_b a - [a, b] \,, \tag{31}$$

will be called the *fundamental torsion extensor field* of $\langle U, \Gamma \rangle$.

- τ is skew-symmetric, i.e.,

$$\tau(b, a) = -\tau(a, b). \tag{32}$$

  Accordingly, there exists a smooth $(2,1)$-*extensor field* on U, defined by

$$\bigwedge^2 \mathcal{V}(U) \ni X^2 \longmapsto \mathcal{T}(X^2) \in \mathcal{V}(U)$$

  such that

$$\mathcal{T}(X^2) = \frac{1}{2} \left\langle \varepsilon^\mu \wedge \varepsilon^\nu, X^2 \right\rangle \tau(e_\mu, e_\nu), \tag{33}$$

  where $\{e_\mu, \varepsilon^\mu\}$ is any pair of dual frame fields on $V \supseteq U$. It should be emphasized that $\mathcal{T}$, as extensor field associated with τ, is well-defined since the vector field $\mathcal{T}(X^2)$ does not depend on the choice of $\{e_\mu, \varepsilon^\mu\}$.

- Such a *torsion extensor field* $\mathcal{T}$ has the basic property

$$\mathcal{T}(a \wedge b) = \tau(a, b). \tag{34}$$

Proof. A straightforward calculation yields

$$\begin{aligned}
\mathcal{T}(a \wedge b) &= \frac{1}{2} \langle \varepsilon^\mu \wedge \varepsilon^\nu, a \wedge b \rangle \tau(e_\mu, e_\nu) = \frac{1}{2} \det \begin{pmatrix} \varepsilon^\mu(a) & \varepsilon^\mu(b) \\ \varepsilon^\nu(a) & \varepsilon^\nu(b) \end{pmatrix} \tau(e_\mu, e_\nu) \\
&= \frac{1}{2} \left(\varepsilon^\mu(a) \varepsilon^\nu(b) - \varepsilon^\mu(b) \varepsilon^\nu(a) \right) \tau(e_\mu, e_\nu) = \frac{1}{2} \left(\tau(a, b) - \tau(b, a) \right),
\end{aligned}$$

and, by taking into account Eq.(32), the expected result follows. ■

It should be remarked that the well known *torsion tensor field* is just given by

$$\mathcal{V}(U)\times\mathcal{V}(U)\times\mathcal{V}^{*}(U)\ni(a,b,\omega)\longmapsto T(a,b,\omega)\in\mathcal{S}(U)$$

such that

$$T(a,b,\omega)=\langle\omega,\tau(a,b)\rangle\,. \tag{35}$$

Note that it is possible to get τ in terms of T, i.e.,

$$\tau(a,b)=T(a,b,\varepsilon^{\mu})e_{\mu}, \tag{36}$$

where $\{e_{\mu},\varepsilon^{\mu}\}$ is any pair of dual frame fields on $V\supseteq U$.

It is also possible to introduce a *third torsion extensor field* for $\langle U,\Gamma\rangle$ by defining the smooth $(1,2)$-*extensor field* on U,

$$V^{*}(U)\ni\omega\longmapsto\Theta(\omega)\in\bigwedge^{2}V^{*}(U),$$

such that

$$\Theta(\omega)=\frac{1}{2}\left\langle\omega,\tau(e_{\mu},e_{\nu})\right\rangle\varepsilon^{\mu}\wedge\varepsilon^{\nu}, \tag{37}$$

where $\{e_{\mu},\varepsilon^{\mu}\}$ is any pair of dual frame fields on $V\supseteq U$. We call Θ the *Cartan torsion extensor field* of $\langle U,\Gamma\rangle$.

The choice of this name is seem to be appropriate once we note that the so-called *Cartan torsion biforms* associated with some pair of dual frame fields $\{e_{\mu},\varepsilon^{\mu}\}$, usually denoted by Θ^{ν}, are just the values of Θ evaluate for ε^{ν}, i.e.,

$$\Theta^{\nu}=\Theta(\varepsilon^{\nu})=\frac{1}{2}\left\langle\varepsilon^{\nu},\tau(e_{\alpha},e_{\beta})\right\rangle\varepsilon^{\alpha}\wedge\varepsilon^{\beta}=\frac{1}{2}T(e_{\alpha},e_{\beta},\varepsilon^{\nu})\varepsilon^{\alpha}\wedge\varepsilon^{\beta}=\frac{1}{2}T_{\alpha\beta}^{\nu}\varepsilon^{\alpha}\wedge\varepsilon^{\beta}.$$

Finally, we present and prove a noticeable property:

- $\mathcal{T}$ and Θ are *duality adjoint* to each other, i.e.,

$$\mathcal{T}^{\triangle}=\Theta\text{ and }\Theta^{\triangle}=\mathcal{T}.$$

Proof. Take $X^{2}\in\bigwedge^{2}\mathcal{V}(U)$ and $\omega\in\mathcal{V}^{*}(U)$. We must prove that

$$\left\langle\omega,\mathcal{T}(X^{2})\right\rangle=\left\langle\Theta(\omega),X^{2}\right\rangle.$$

Using Eq.(33) and Eq.(37), we can write that

$$\left\langle \omega, \mathcal{T}(X^2) \right\rangle = \left\langle \omega, \frac{1}{2} \left\langle \varepsilon^\mu \wedge \varepsilon^\nu, X^2 \right\rangle \tau(e_\mu, e_\nu) \right\rangle = \frac{1}{2} \left\langle \varepsilon^\mu \wedge \varepsilon^\nu, X^2 \right\rangle \langle \omega, \tau(e_\mu, e_\nu) \rangle$$
$$= \left\langle \frac{1}{2} \langle \omega, \tau(e_\mu, e_\nu) \rangle \varepsilon^\mu \wedge \varepsilon^\nu, X^2 \right\rangle = \left\langle \Theta(\omega), X^2 \right\rangle,$$

which proves our statement. ■

4.1 Cartan First Equation

- Let $\langle U, \Gamma \rangle$ be a parallelism structure on U. Take any pair of dual frame fields $\{e_\mu, \varepsilon^\mu\}$ on $V \supseteq U$. We have

$$\Theta(\omega) = d\omega - \Gamma^-(e_\sigma, \omega) \wedge \varepsilon^\sigma \tag{38}$$

Proof. A straightforward calculation yields

$$\Theta(\omega) = \frac{1}{2} \left\langle \omega, \nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu - [e_\mu, e_\nu] \right\rangle \varepsilon^\mu \wedge \varepsilon^\nu$$
$$= \left\langle \omega, \nabla_{e_\mu} e_\nu \right\rangle \varepsilon^\mu \wedge \varepsilon^\nu - \frac{1}{2} \langle \omega, [e_\mu, e_\nu] \rangle \varepsilon^\mu \wedge \varepsilon^\nu,$$

but, by recalling the identity

$$d\omega = d\omega(e_\nu) \wedge \varepsilon^\nu - \frac{1}{2} \langle \omega, [e_\mu, e_\nu] \rangle \varepsilon^\mu \wedge \varepsilon^\nu$$

valid for smooth form fields we get

$$\Theta(\omega) = \Gamma^+(e_\nu, \omega) \wedge \varepsilon^\nu + (d\omega - d\omega(e_\nu) \wedge \varepsilon^\nu),$$

from where using Eq.(29) the expected result follows. ■

We emphasis that Eq.(38) is the *frame field independent version* of the so-called *Cartan first structure equation.* In fact, if we choose some pair of dual frame fields $\{e_\mu, \varepsilon^\mu\}$, we can write

$$\Theta(\varepsilon^\nu) = d\varepsilon^\nu - \Gamma^-(e_\sigma, \varepsilon^\nu) \wedge \varepsilon^\sigma,$$

i.e.,

$$\Theta^\nu = d\varepsilon^\nu + \gamma^\nu_\sigma \wedge \varepsilon^\sigma. \tag{39}$$

What the meaning of the second term in Eq.(38)?[3]

A straightforward calculation gives

$$\begin{aligned}\Gamma^-(e_\sigma,\omega)\wedge\varepsilon^\sigma &= \left\langle \nabla_{e_\mu}\omega, e_\sigma\right\rangle \wedge\varepsilon^\sigma = \varepsilon^\mu\wedge\nabla_{e_\mu}\omega,\\ \Gamma^-(e_\sigma,\omega)\wedge\varepsilon^\sigma &= \nabla\wedge\omega.\end{aligned} \tag{40}$$

Thus, the second term in Eq.(38) is just the[10] *covariant curl* $(\nabla\wedge\omega)$ of the smooth form field ω. When $\Theta=0$ we have the identity

$$d\omega = \nabla\wedge\omega. \tag{41}$$

5 Curvature

Let $\langle U,\Gamma\rangle$ be a parallelism structure on U. The smooth 3-*covariant extensor vector field* on U, defined by

$$\mathcal{V}(U)\times\mathcal{V}(U)\times\mathcal{V}(U)\ni(a,b,c)\longmapsto\rho(a,b,c)\in\mathcal{V}(U),$$

such that

$$\rho(a,b,c)=[\nabla_a,\nabla_b]\,c-\nabla_{[a,b]}c, \tag{42}$$

will be called the *fundamental curvature extensor field* of $\langle U,\Gamma\rangle$.

- As can be easily verified, ρ is skew-symmetric with respect to the first and the second variables, i.e.,

$$\rho(b,a,c)=-\rho(a,b,c). \tag{43}$$

Thus, there exists a smooth 1 *bivector and* 1 *vector variables vector extensor field* on U, defined by

$$\bigwedge^2\mathcal{V}(U)\times\mathcal{V}(U)\ni(X^2,c)\longmapsto\mathcal{R}(X^2,c)\in\mathcal{V}(U)$$

such that

$$\mathcal{R}(X^2,c)=\frac{1}{2}\left\langle\varepsilon^\mu\wedge\varepsilon^\nu,X^2\right\rangle\rho(e_\mu,e_\nu,c), \tag{44}$$

where $\{e_\mu,\varepsilon^\mu\}$ is any pair of dual frame fields on $V\supseteq U$. We note that $\mathcal{R}$, as extensor field associated with ρ, is well-defined since the vector field $\mathcal{R}(X^2,c)$ does not depend on the choice of $\{e_\mu,\varepsilon^\mu\}$.

[3]Note that the strong linearity of Γ^- is essential in warranting that $\Gamma^-(e_\sigma,\omega)\wedge\varepsilon^\sigma$ has frame field independent character, i.e., that it does not depend on the choice of $\{e_\mu,\varepsilon^\mu\}$.

- The *curvature extensor field* $\mathcal{R}$ has the basic property

$$\mathcal{R}(a \wedge b, c) = \rho(a, b, c). \tag{45}$$

Proof. A straightforward calculation gives

$$\begin{aligned}
\mathcal{R}(a \wedge b, c) &= \frac{1}{2} \left\langle \varepsilon^{\mu} \wedge \varepsilon^{\nu}, a \wedge b \right\rangle \rho(e_{\mu}, e_{\nu}, c) \\
&= \frac{1}{2} \det \begin{pmatrix} \varepsilon^{\mu}(a) & \varepsilon^{\mu}(b) \\ \varepsilon^{\nu}(a) & \varepsilon^{\nu}(b) \end{pmatrix} \rho(e_{\mu}, e_{\nu}, c) \\
&= \frac{1}{2} \left(\varepsilon^{\mu}(a)\varepsilon^{\nu}(b) - \varepsilon^{\mu}(b)\varepsilon^{\nu}(a)\right) \rho(e_{\mu}, e_{\nu}, c) \\
&= \frac{1}{2} \left(\rho(a, b, c) - \rho(b, a, c)\right),
\end{aligned}$$

and, by using Eq.(43), we get the expected result. ■

We remark that the so-called *curvature tensor field* is just given by

$$\mathcal{V}(U) \times \mathcal{V}(U) \times \mathcal{V}(U) \times V^{*}(U) \ni (a, b, c, \omega) \longmapsto R(a, b, c, \omega) \in \mathcal{S}(U)$$

such that

$$R(a, b, c, \omega) = \left\langle \omega, \rho(b, c, a) \right\rangle. \tag{46}$$

It is possible to get ρ in terms of R, i.e.,

$$\rho(a, b, c) = R(c, a, b, \varepsilon^{\mu}) e_{\mu}, \tag{47}$$

where $\{e_{\mu}, \varepsilon^{\mu}\}$ is any frame field on $V \supseteq U$.

We note that it is also possible to introduce a *third curvature extensor field* for $\langle U, \Gamma \rangle$ defining the smooth 1 *vector and* 1 *form variables form extensor field* on U

$$\mathcal{V}(U) \times \mathcal{V}^{*}(U) \ni (c, \omega) \longmapsto \Omega\,(c, \omega) \in \bigwedge^{2} V^{*}(U),$$

such that

$$\Omega\,(c, \omega) = \frac{1}{2} \left\langle \omega, \rho(e_{\mu}, e_{\nu}, c) \right\rangle \varepsilon^{\mu} \wedge \varepsilon^{\nu}, \tag{48}$$

where $\{e_{\mu}, \varepsilon^{\mu}\}$ is any pair of dual frame fields on $V \supseteq U$. We call Ω the *Cartan curvature extensor field* of $\langle U, \Gamma \rangle$ since the so-called *Cartan*

curvature biforms associated with some $\{e_\mu, \varepsilon^\mu\}$, usually denoted by Ω_μ^ν, are exactly the values of Ω evaluated for (e_μ, ε^ν), i.e.

$$\Omega_\mu^\nu := \Omega\left(e_\mu, \varepsilon^\nu\right) = \frac{1}{2}\left\langle \varepsilon^\nu, \rho(e_\alpha, e_\beta, e_\mu)\right\rangle \varepsilon^\alpha \wedge \varepsilon^\beta = \frac{1}{2}R(e_\mu, e_\alpha, e_\beta, \varepsilon^\nu)\varepsilon^\alpha \wedge \varepsilon^\beta.$$

Finally, we present and prove a remarkable property:

- $\mathcal{R}_c$ and Ω_c are *duality adjoint* of each other, i.e.,

$$\mathcal{R}_c^\triangle = \Omega_c \text{ and } \Omega_c^\triangle = \mathcal{R}_c. \tag{49}$$

Proof. Let us take $X^2 \in \bigwedge^2 \mathcal{V}(U)$ and $\omega \in V^*(U)$. we have to prove that

$$\left\langle \omega, \mathcal{R}_c(X^2)\right\rangle = \left\langle \Omega_c(\omega), X^2\right\rangle.$$

By using Eq.(44) and Eq.(48), we in fact can write that

$$\begin{aligned}
\left\langle \omega, \mathcal{R}_c(X^2)\right\rangle &= \left\langle \omega, \frac{1}{2}\left\langle \varepsilon^\mu \wedge \varepsilon^\nu, X^2\right\rangle \rho(e_\mu, e_\nu, c)\right\rangle \\
&= \frac{1}{2}\left\langle \varepsilon^\mu \wedge \varepsilon^\nu, X^2\right\rangle \left\langle \omega, \rho(e_\mu, e_\nu, c)\right\rangle \\
&= \left\langle \frac{1}{2}\left\langle \omega, \rho(e_\mu, e_\nu, c)\right\rangle \varepsilon^\mu \wedge \varepsilon^\nu, X^2\right\rangle = \left\langle \Omega_c(\omega), X^2\right\rangle.
\end{aligned}$$

■

5.1 Cartan Second Equation

- Let $\langle U, \Gamma\rangle$ be a parallelism structure on U. Let us take any pair of dual frame fields $\{e_\mu, \varepsilon^\mu\}$ on $V \supseteq U$. We have

$$\Omega(c, \omega) = d\Gamma^+(c, \omega) + \Gamma^+(c, \varepsilon^\sigma) \wedge \Gamma^-(e_\sigma, \omega). \tag{50}$$

Proof. A straightforward calculation yields

$$\begin{aligned}
\Omega(c, \omega) &= \frac{1}{2}\left\langle \omega, \left[\nabla_{e_\mu}, \nabla_{e_\nu}\right] c - \nabla_{[e_\mu, e_\nu]}c\right\rangle \varepsilon^\mu \wedge \varepsilon^\nu \\
&= \left\langle \omega, \nabla_{e_\mu}\nabla_{e_\nu}c\right\rangle \varepsilon^\mu \wedge \varepsilon^\nu - \frac{1}{2}\left\langle \omega, \nabla_{[e_\mu, e_\nu]}c\right\rangle \varepsilon^\mu \wedge \varepsilon^\nu, \qquad \text{(a)}
\end{aligned}$$

By using Eq.(23), the first term in (a) gives

$$\begin{aligned}\left\langle\omega,\nabla_{e_\mu}\nabla_{e_\nu}c\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu &= \left\langle\omega,\nabla_{e_\mu}\left\langle\Gamma^+(c,\varepsilon^\sigma),e_\nu\right\rangle e_\sigma\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu\\ &= \left\langle\omega,e_\mu\left\langle\Gamma^+(c,\varepsilon^\sigma),e_\nu\right\rangle e_\sigma\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu\\ &+ \left\langle\omega,\left\langle\Gamma^+(c,\varepsilon^\sigma),e_\nu\right\rangle\nabla_{e_\mu}e_\sigma\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu\\ &= \left(e_\mu\left\langle\Gamma^+(c,\varepsilon^\sigma),e_\nu\right\rangle\right)\langle\omega,e_\sigma\rangle\varepsilon^\mu\wedge\varepsilon^\nu\\ &+ \left\langle\Gamma^+(c,\varepsilon^\sigma),e_\nu\right\rangle\left\langle\omega,\nabla_{e_\mu}e_\sigma\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu, \qquad \text{(b)}\end{aligned}$$

but, the first term in (b) and the second term in (b) can be written

$$\begin{aligned}&\left(e_\mu\left\langle\Gamma^+(c,\varepsilon^\sigma),e_\nu\right\rangle\right)\langle\omega,e_\sigma\rangle\varepsilon^\mu\wedge\varepsilon^\nu\\ &= d\left\langle\Gamma^+(c,\omega),e_\nu\right\rangle\wedge\varepsilon^\nu - d\langle\omega,e_\sigma\rangle\wedge\Gamma^+(c,\varepsilon^\sigma) \qquad \text{(c)}\end{aligned}$$

and

$$\left\langle\Gamma^+(c,\varepsilon^\sigma),e_\nu\right\rangle\left\langle\omega,\nabla_{e_\mu}e_\sigma\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu = \Gamma^+(e_\sigma,\omega)\wedge\Gamma^+(c,\varepsilon^\sigma). \qquad \text{(d)}$$

Thus, by putting (c) and (d) into (b), and by taking into account Eq.(29), we get

$$\left\langle\omega,\nabla_{e_\mu}\nabla_{e_\nu}c\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu = d\left\langle\Gamma^+(c,\omega),e_\nu\right\rangle\wedge\varepsilon^\nu - \Gamma^-(e_\sigma,\omega)\wedge\Gamma^+(c,\varepsilon^\sigma). \quad \text{(e)}$$

By using once again Eq.(23), the second term in (a) gives

$$\begin{aligned}\frac{1}{2}\left\langle\omega,\nabla_{[e_\mu,e_\nu]}c\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu &= \frac{1}{2}\left\langle\omega,\left\langle\Gamma^+(c,\varepsilon^\sigma),[e_\mu,e_\nu]\right\rangle e_\sigma\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu\\ &= \frac{1}{2}\left\langle\Gamma^+(c,\varepsilon^\sigma),[e_\mu,e_\nu]\right\rangle\langle\omega,e_\sigma\rangle\varepsilon^\mu\wedge\varepsilon^\nu\\ &= \frac{1}{2}\left\langle\Gamma^+(c,\omega),[e_\mu,e_\nu]\right\rangle\varepsilon^\mu\wedge\varepsilon^\nu. \qquad \text{(f)}\end{aligned}$$

By putting (e) and (f) into (a), and recalling the identity for smooth form fields $d\sigma = d\sigma(e_\nu)\wedge\varepsilon^\nu - \frac{1}{2}\langle\sigma,[e_\mu,e_\nu]\rangle\varepsilon^\mu\wedge\varepsilon^\nu$, we get the expected result. ■

We emphasis that Eq.(50) is the frame field independent version of the so-called Cartan second structure equation. Indeed, if we choose some pair of dual frame fields $\{e_\mu,\varepsilon^\mu\}$, we can write

$$\Omega(e_\mu,\varepsilon^\nu) = d\Gamma^+(e_\mu,\varepsilon^\nu) + \Gamma^+(e_\mu,\varepsilon^\sigma)\wedge\Gamma^-(e_\sigma,\varepsilon^\nu),$$

i.e.,

$$\Omega_\mu^\nu = d\gamma_\mu^\nu + \gamma_\sigma^\nu \wedge \omega_\mu^\sigma. \tag{51}$$

What is the meaning of the second term in Eq(50)?[4]

The answer is given by

$$\begin{aligned}
\Gamma^+(c,\varepsilon^\sigma) \wedge \Gamma^-(e_\sigma,\omega) &= \left\langle \varepsilon^\sigma, \nabla_{e_\mu} c \right\rangle \left\langle \nabla_{e_\nu}\omega, e_\sigma \right\rangle \varepsilon^\mu \wedge \varepsilon^\nu, \\
\Gamma^+(c,\varepsilon^\sigma) \wedge \Gamma^-(e_\sigma,\omega) &= \left\langle \nabla_{e_\nu}\omega, \nabla_{e_\mu} c \right\rangle \varepsilon^\mu \wedge \varepsilon^\nu.
\end{aligned} \tag{52}$$

6 Symmetric Parallelism Structure

A parallelism structure $\langle U, \Gamma \rangle$ is said to be *symmetric* if and only if for all smooth vector fields a and b it holds

$$\Gamma(a,b) - \Gamma(b,a) = [a,b]\,, \tag{53}$$

i.e.,

$$\nabla_a b - \nabla_b a = [a,b]\,. \tag{54}$$

Now, according with Eq.(31), we see that the *condition of symmetry* is completely equivalent to the *condition of null torsion*, i.e.,

$$\tau(a,b) = 0. \tag{55}$$

So, taking into account Eq.(33) and Eq.(37), we also have that

$$\mathcal{T}(X^2) = 0 \text{ and } \Theta(\omega) = 0. \tag{56}$$

We present and prove two noticeable properties for a symmetric parallelism structure.

- The fundamental curvature field ρ satisfies a *cyclic property*, i.e.,

$$\rho(a,b,c) + \rho(b,c,a) + \rho(c,a,b) = 0.$$

[4]Observe that the strong linearities of Γ^+ and Γ^- are essential for making $\Gamma^+(c,\varepsilon^\sigma) \wedge \Gamma^-(e_\sigma,\omega)$ frame field independent.

Proof. Let us take $a, b, c \in \mathcal{V}(U)$. By using Eq.(42), we can write

$$\rho(a,b,c) = \nabla_a \nabla_b c - \nabla_b \nabla_a c - \nabla_{[a,b]} c, \tag{a}$$
$$\rho(b,c,a) = \nabla_b \nabla_c a - \nabla_c \nabla_b a - \nabla_{[b,c]} a, \tag{b}$$
$$\rho(c,a,b) = \nabla_c \nabla_a b - \nabla_a \nabla_c b - \nabla_{[c,a]} b. \tag{c}$$

By adding (a), (b) and (c), we have

$$\begin{aligned} &\rho(a,b,c) + \rho(b,c,a) + \rho(c,a,b) \\ &= \nabla_a(\nabla_b c - \nabla_c b) + \nabla_b(\nabla_c a - \nabla_a c) + \nabla_c(\nabla_a b - \nabla_b a) \\ &- \nabla_{[a,b]} c - \nabla_{[b,c]} a - \nabla_{[c,a]} b, \end{aligned} \tag{d}$$

but, by taking into account Eq.(54), we get

$$\rho(a,b,c) + \rho(b,c,a) + \rho(c,a,b) = [a,[b,c]] + [b,[c,a]] + [c,[a,b]], \tag{e}$$

whence, by recalling the Jacobi identities for the Lie product of smooth vector fields, the expected result immediately follows. ■

- The fundamental curvature field ρ satisfies the so-called Bianchi *identity*, i.e.,

$$\nabla_w \rho(a,b,c) + \nabla_a \rho(b,w,c) + \nabla_b \rho(w,a,c) = 0.$$

Note the cycling of letters: $a, b, w \to b, w, a \to w, a, b$.

Proof. Let us take $a, b, c, w \in \mathcal{V}(U)$. By using Eq.(16), we have

$$\begin{aligned} &\nabla_w \rho(a,b,c) \\ &= \nabla_w(\rho(a,b,c)) - \rho(\nabla_w a, b, c) - \rho(a, \nabla_w b, c) - \rho(a, b, \nabla_w c), \end{aligned} \tag{a}$$
$$\begin{aligned} &\nabla_a \rho(b,w,c) \\ &= \nabla_a(\rho(b,w,c)) - \rho(\nabla_a b, w, c) - \rho(b, \nabla_a w, c) - \rho(b, w, \nabla_a c), \end{aligned} \tag{b}$$
$$\begin{aligned} &\nabla_b \rho(w,a,c) \\ &= \nabla_b(\rho(w,a,c)) - \rho(\nabla_b w, a, c) - \rho(w, \nabla_b a, c) - \rho(w, a, \nabla_b c). \end{aligned} \tag{c}$$

By adding (a), (b) and (c), and by taking into account Eq.(43) and Eq.(54), we get

$$\begin{aligned} &\nabla_w \rho(a,b,c) + \nabla_a \rho(b,w,c) + \nabla_b \rho(w,a,c) \\ &= \nabla_w(\rho(a,b,c)) + \nabla_a(\rho(b,w,c)) + \nabla_b(\rho(w,a,c)) \\ &- \rho([w,a], b, c) - \rho([a,b], w, c) - \rho([b,w], a, c) \\ &- \rho(a, b, \nabla_w c) - \rho(b, w, \nabla_a c) - \rho(w, a, \nabla_b c). \end{aligned} \tag{57}$$

By using Eq.(42), the first term of (d) can be written

$$\begin{aligned}
&\nabla_w(\rho(a,b,c)) + \nabla_a(\rho(b,w,c)) + \nabla_b(\rho(w,a,c)) \\
&= \nabla_w\left([\nabla_a, \nabla_b]\, c\right) + \nabla_a\left([\nabla_b, \nabla_w]\, c\right) + \nabla_b\left([\nabla_w, \nabla_a]\, c\right) \\
&- \nabla_w \nabla_{[a,b]} c - \nabla_a \nabla_{[b,w]} c - \nabla_b \nabla_{[w,a]} c.
\end{aligned} \tag{58}$$

By using Eq.(42) and by recalling the so-called Jacobi identity for the Lie product of smooth vector fields, the second term of (d) can be written

$$\begin{aligned}
&- \rho([w,a]\,, b, c) - \rho([a,b]\,, w, c) - \rho([b,w]\,, a, c) \\
&= -\nabla_{[w,a]} \nabla_b c - \nabla_{[a,b]} \nabla_w c - \nabla_{[b,w]} \nabla_a c \\
&+ \nabla_b \nabla_{[w,a]} c + \nabla_w \nabla_{[a,b]} c + \nabla_a \nabla_{[b,w]} c.
\end{aligned} \tag{f}$$

By adding (e) and (f), we get

$$\begin{aligned}
&\nabla_w(\rho(a,b,c)) + \nabla_a(\rho(b,w,c)) + \nabla_b(\rho(w,a,c)) \\
&- \rho([w,a]\,, b, c) - \rho([a,b]\,, w, c) - \rho([b,w]\,, a, c) \\
&= \nabla_w\left([\nabla_a, \nabla_b]\, c\right) + \nabla_a\left([\nabla_b, \nabla_w]\, c\right) + \nabla_b\left([\nabla_w, \nabla_a]\, c\right) \\
&- \nabla_{[w,a]} \nabla_b c - \nabla_{[a,b]} \nabla_w c - \nabla_{[b,w]} \nabla_a c,
\end{aligned} \tag{59}$$

now, by sustaining the Eq. (59) in the Eq. (57), and by using once again Eq.(42), we get

$$\begin{aligned}
&\nabla_w(\rho(a,b,c)) + \nabla_a(\rho(b,w,c)) + \nabla_b(\rho(w,a,c)) \\
&= \nabla_w\left([\nabla_a, \nabla_b]\, c\right) + \nabla_a\left([\nabla_b, \nabla_w]\, c\right) + \nabla_b\left([\nabla_w, \nabla_a]\, c\right) \\
&-\nabla_{[w,a]} \nabla_b c - \nabla_{[a,b]} \nabla_w c - \nabla_{[b,w]} \nabla_a c - [\nabla_a, \nabla_b]\, \nabla_w c + \nabla_{[a,b]} \nabla_w c \\
&- [\nabla_b, \nabla_w]\, \nabla_a c + \nabla_{[b,w]} \nabla_a c - [\nabla_w, \nabla_a]\, \nabla_b c + \nabla_{[w,a]} \nabla_b c \\
&= \left\{[\nabla_w, [\nabla_a, \nabla_b]] + [\nabla_a, [\nabla_b, \nabla_w]] + [\nabla_b, [\nabla_w, \nabla_a]]\right\} c
\end{aligned} \tag{60}$$

by recalling the so-called Jacobi identities for the Lie product of smooth vector fields, the expected result immediately follows. ■

7 Deformed Parallelism Structure

Let $\langle U, \Gamma \rangle$ be a parallelism structure on U. Let us take an invertible smooth extensor operator field λ on $V \supseteq U$, i.e., $\lambda : \mathcal{V}(U) \rightarrow \mathcal{V}(U)$. We can construct another well-defined connection on U, namely $\overset{\lambda}{\Gamma}$, given by

$$\mathcal{V}(U) \times \mathcal{V}(U) \ni (a, v) \longmapsto \overset{\lambda}{\Gamma}(a, v) \in \mathcal{V}(U)$$

such that

$$\overset{\lambda}{\Gamma}(a,v)=\lambda(\Gamma(a,\lambda^{-1}(v))). \tag{61}$$

$\overset{\lambda}{\Gamma}$ is indeed a connection on U since it satisfies Eq.(1) and Eq.(2). It will be called the λ-*deformation* of Γ.

The parallelism structure $\left\langle U,\overset{\lambda}{\Gamma}\right\rangle$ is said to be the λ-*deformation* of $\langle U,\Gamma\rangle$.

Let us take $a\in\mathcal{V}(U)$. The *a-Directional Covariant Derivative Operator* (*a-DCDO*) associated with $\left\langle U,\overset{\lambda}{\Gamma}\right\rangle$, namely $\overset{\lambda}{\nabla}_a$, has the basic properties:

- For all $v\in\mathcal{V}(U)$

  $$\overset{\lambda}{\nabla}_a v=\lambda(\nabla_a\lambda^{-1}(v)). \tag{62}$$

  It follows from Eq.(3) and Eq.(61).

- For all $\omega\in\mathcal{V}^*(U)$

  $$\overset{\lambda}{\nabla}_a\omega=\lambda^{-\vartriangle}(\nabla_a\lambda^{\vartriangle}(\omega)). \tag{63}$$

Proof. Let us take $v\in\mathcal{V}(U)$. By using Eq.(4) and Eq.(62), we have

$$\begin{aligned}\overset{\lambda}{\nabla}_a\omega(v)&=a\omega(v)-\omega(\overset{\lambda}{\nabla}_a v)\\&=a\omega(v)-\omega(\lambda(\nabla_a\lambda^{-1}(v)))\\&=a\langle\omega,v\rangle-\left\langle\omega,\lambda(\nabla_a\lambda^{-1}(v))\right\rangle,\end{aligned} \tag{a}$$

but, by recalling the fundamental property of the *duality adjoint* and by using once again Eq.(4), the second term in (a) can be written

$$\begin{aligned}\left\langle\omega,\lambda(\nabla_a\lambda^{-1}(v))\right\rangle&=\left\langle\lambda^{\vartriangle}(\omega),\nabla_a\lambda^{-1}(v)\right\rangle\\&=a\left\langle\lambda^{\vartriangle}(\omega),\lambda^{-1}(v)\right\rangle-\left\langle\nabla_a\lambda^{\vartriangle}(\omega),\lambda^{-1}(v)\right\rangle\\&=a\langle\omega,v\rangle-\left\langle\lambda^{-\vartriangle}\nabla_a\lambda^{\vartriangle}(\omega),v\right\rangle,\end{aligned} \tag{b}$$

Finally, putting (b) into (a), the expected result follows. ■

8 Relative Parallelism Structure

Let again $\{b_\mu, \beta^\mu\}$ be a pair of dual frame fields for $U \subseteq M$. Associated with $\{b_\mu, \beta^\mu\}$ we can construct a well-defined connection on U given by the mapping

$$B : \mathcal{V}(U) \times \mathcal{V}(U) \longrightarrow \mathcal{V}(U),$$

such that

$$B(a, v) = [a\beta^\sigma(v)]\, b_\sigma. \tag{64}$$

B is called *relative connection* on U with respect to $\{b_\mu, \beta^\mu\}$ (or simply relative connection for short).

The parallelism structure $\langle U, B\rangle$ will be called *relative parallelism structure* with respect to $\{b_\mu, \beta^\mu\}$.

The *a-DCDO* induced by the relative connection will be denoted by ∂_a. According with Eq.(3), the *a-DCD* of a smooth vector field is given by

$$\partial_a v = [a\beta^\sigma(v)]\, b_\sigma. \tag{65}$$

- Note that ∂_a is the unique *a-DCDO* which satisfies the condition

$$\partial_a b_\mu = 0. \tag{66}$$

From Eq.(4) and Eq.(66), the *a-DCD* of a smooth form field is given by

$$\partial_a \omega = [a\omega(b_\sigma)]\, \beta^\sigma. \tag{67}$$

Then, it holds also

- $$\partial_a \beta^\nu = 0. \tag{68}$$

The relative parallelism structure has the basic properties:

- The fundamental torsion extensor field and, the Cartan torsion extensor field are given by

$$\tau(a, b) = [d\beta^\sigma(a, b)]\, b_\sigma, \tag{69}$$

$$\Theta(\omega) = \langle \omega, b_\sigma\rangle\, d\beta^\sigma. \tag{70}$$

- The fundamental curvature extensor field of $\langle U, B\rangle$ vanishes, i.e., $\langle U, B\rangle$ is such that
$$\rho(a,b,c)=0. \tag{71}$$

We present only the proof of the properties given by the Eqs. (69) and (70), the other proofs are analogous.

Proof. a) First note that the fundamental torsion extensor field, associated with the parallelism structure $\langle U, B\rangle$, is defined by

$$\tau\left(a,b\right)=\partial_a b-\partial_b a-\left[a,b\right],$$

then by using the Eq.(67), we can write

$$\begin{aligned}
\tau\left(a,b\right) &= \partial_a b-\partial_b a-\left[a,b\right]\\
&= \left[a\beta^{\sigma}\left(b\right)\right]b_{\sigma}-\left[b\beta^{\sigma}\left(a\right)\right]b_{\sigma}-\varepsilon^{\mu}\left(a\right)\varepsilon^{\nu}\left(b\right)c_{\mu\nu}^{\sigma}b_{\sigma}\\
&= \left[a\beta^{\sigma}\left(b\right)-b\beta^{\sigma}\left(a\right)-\varepsilon^{\mu}\left(a\right)\varepsilon^{\nu}\left(b\right)c_{\mu\nu}^{\sigma}\right]b_{\sigma},
\end{aligned} \tag{72}$$

where $\left[a,b\right]=\left[\varepsilon^{\mu}\left(a\right)b_{\mu},\varepsilon^{\nu}\left(b\right)b_{\nu}\right]=\varepsilon^{\mu}\left(a\right)\varepsilon^{\nu}\left(b\right)\left[b_{\mu},b_{\nu}\right]=\varepsilon^{\mu}\left(a\right)\varepsilon^{\nu}\left(b\right)c_{\mu\nu}^{\sigma}b_{\sigma}$.

On the other hand, if $\mathcal{L}_a S$ denote the Lie derivative of S in the direction of a, (see, e.g., [1, 10]), we have

$$\begin{aligned}
d\beta^{\sigma}\left(a,b\right) &= \mathcal{L}_a\left(\beta^{\sigma}\left(b\right)\right)-\mathcal{L}_b\left(\beta^{\sigma}\left(a\right)\right)-\varepsilon^{\sigma}\left(\left[a,b\right]\right)\\
&= a\left(\beta^{\sigma}\left(b\right)\right)-b\left(\beta^{\sigma}\left(a\right)\right)-\varepsilon^{\mu}\left(a\right)\varepsilon^{\nu}\left(b\right)c_{\mu\nu}^{\sigma}.
\end{aligned} \tag{73}$$

Thus, from the Eq. (72) and (73) the result follows.

b) Now, for proofing Eq. (70), note that from Eq. (68) we can write

$$\tau\left(b_{\mu},b_{\nu}\right)=\partial_{b_{\mu}}b_{\nu}-\partial_{b_{\nu}}b_{\mu}-\left[b_{\mu},b_{\nu}\right]=-\left[b_{\mu},b_{\nu}\right],$$

and by definition we have $\Theta\left(\omega\right)=\frac{1}{2}\left\langle\omega,\tau\left(b_{\mu},b_{\nu}\right)\right\rangle\beta^{\mu}\wedge\beta^{\nu}$. Then,

$$\begin{aligned}
\Theta\left(\omega\right) &= -\tfrac{1}{2}\left\langle\omega,\left[b_{\mu},b_{\nu}\right]\right\rangle\beta^{\mu}\wedge\beta^{\nu}\\
&= -\tfrac{1}{2}\left\langle\omega\left(b_{\sigma}\right)\beta^{\sigma},\left[b_{\mu},b_{\nu}\right]\right\rangle\beta^{\mu}\wedge\beta^{\nu}\\
&= \omega\left(b_{\sigma}\right)\left(-\tfrac{1}{2}\right)\left\langle\beta^{\sigma},\left[b_{\mu},b_{\nu}\right]\right\rangle\beta^{\mu}\wedge\beta^{\nu}.
\end{aligned} \tag{74}$$

On the other hand,

$$\begin{aligned}
d\beta^{\sigma} &= d\beta^{\sigma}\left(b_{\nu}\right)\wedge\beta^{\nu}-\tfrac{1}{2}\left\langle\beta^{\sigma},\left[b_{\mu},b_{\nu}\right]\right\rangle\beta^{\mu}\wedge\beta^{\nu}\\
&= -\tfrac{1}{2}\left\langle\beta^{\sigma},\left[b_{\mu},b_{\nu}\right]\right\rangle\beta^{\mu}\wedge\beta^{\nu},
\end{aligned} \tag{75}$$

thus, from Eqs. (74) and (75) the result follows. ■

8.1 Split Theorem

Let $\langle U_0, \Gamma\rangle$ be a parallelism structure on U_0. Let us take any relative parallelism structures $\langle U, B\rangle$ such that $U_0 \cap U \neq \emptyset$. There exists a smooth 2-*covariant vector extensor field* on $U_0 \cap U$, namely γ, defined by

$$\mathcal{V}(U_0 \cap U) \times \mathcal{V}(U_0 \cap U) \ni (a, v) \longmapsto \gamma(a, v) \in \mathcal{V}(U_0 \cap U)$$

such that

$$\gamma(a, v) = \beta^\mu(v)\nabla_a b_\mu \tag{76}$$

which satisfies

$$\Gamma(a, v) = B(a, v) + \gamma(a, v). \tag{77}$$

Such a extensor field γ will be called the *relative connection extensor field*[5] on $U_0 \cap U$.

From Eq.(3),this means that for all $v \in \mathcal{V}(U_0 \cap U)$:

$$\nabla_a v = \partial_a v + \gamma_a(v), \tag{78}$$

where ∇_a is the *a-DCDO* associated with $\langle U_0, \Gamma\rangle$ and ∂_a is the *a-DCDO* associated with $\langle U, B\rangle$ (note that γ_a is a smooth *vector operator field* on $U_0 \cap U$ defined by $\gamma_a(v) = \gamma(a, v)$).

By using Eq.(4) and Eq.(78), we get that for all $\omega \in \mathcal{V}^*(U_0 \cap U)$:

$$\nabla_a \omega = \partial_a \omega - \gamma_a^\triangle(\omega), \tag{79}$$

where $\gamma_a^\triangle$ is the *dual adjoint* of γ_a (i.e., $\left\langle \gamma_a^\triangle(\omega), v\right\rangle = \langle \omega, \gamma_a(v)\rangle$).

8.2 Jacobian Fields

Let $\{b_\mu, \beta^\mu\}$ and $\left\{b'_\mu, \beta^{\mu\prime}\right\}$ be any two pairs of dual frame fields on the open sets $U \subseteq M$ and $U' \subseteq M$, respectively.

If the parallelism structure $\langle U, B\rangle$ is compatible with the parallelism structure $\langle U', B'\rangle$ (i.e., define the same connection on $U \cap U' \neq \emptyset$), then we can define a smooth *extensor operator field* on $U \cap U'$, namely J, by

$$\mathcal{V}(U \cap U') \ni v \longmapsto J(v) \in \mathcal{V}(U \cap U'),$$

[5]The properties of the tensor field $\gamma^\alpha_{\mu\nu}$ such that $\gamma(\partial_\mu, \partial_\nu) = \gamma^\alpha_{\mu\nu}\partial_\alpha$ where $\{\partial_\mu\}$ is a basis for $\mathcal{V}(U_0 \cap U)$ are studied in details in [10].

such that

$$J(v) = \beta^{\sigma}(v)b'_{\sigma}. \tag{80}$$

It will be called the *Jacobian field* associated with the pairs of frame fields $\{b_{\mu}, \beta^{\mu}\}$ and $\{b'_{\mu}, \beta^{\mu\prime}\}$ (in this order!).

Note that in accordance with the above definition the Jacobian field associated with $\{b'_{\mu}, \beta^{\mu\prime}\}$ and $\{b_{\mu}, \beta^{\mu}\}$ is J', given by

$$\mathcal{V}(U \cap U') \ni v \longmapsto J'(v) \in \mathcal{V}(U \cap U'),$$

such that

$$J'(v) = \beta^{\sigma\prime}(v)b_{\sigma}. \tag{81}$$

It is the *inverse extensor operator* of J, i.e., $J \circ J'(v) = v$ and $J' \circ J(v) = v$ for each $v \in \mathcal{V}(U \cap U')$.

We note that

$$J(b_{\mu}) = b'_{\mu} \text{ and } J^{-1}(b'_{\mu}) = b_{\mu}. \tag{82}$$

- Take $a \in \mathcal{V}(U \cap U')$, the *a-DCDO* associated with $\langle U, B\rangle$ and $\langle U', B'\rangle$, namely ∂_a and ∂'_a, are related by

$$\partial'_a v = J(\partial_a J^{-1}(v)). \tag{83}$$

Proof. A straightforward calculation, yields

$$\partial'_a v = (a\beta^{\mu\prime}(v))b'_{\mu} = (a\beta^{\mu\prime}(v))J(b_{\mu}).$$

Using the identity $\beta^{\mu}(J^{-1}(v)) = \beta^{\mu\prime}(v)$ (valid for smooth extensor fields) we get

$$\partial'_a v = J((a\beta^{\mu}(J^{-1}(v))b_{\mu}),$$

from where the expected result follows. ■

We see that from the definition of deformed parallelism structure, ∂'_a is a *J-deformation* of ∂_a.

Then, from Eq.(63), we have

$$\partial'_a \omega = J^{-\triangle}(\partial_a J^{\triangle}(\omega)). \tag{84}$$

Finally, we note that

$$J^{-\triangle}(\beta^{\mu}) = \beta^{\mu\prime} \text{ and } J^{\triangle}(\beta^{\mu\prime}) = \beta^{\mu}. \tag{85}$$

9 Conclusions

In this paper using the algebra of extensor fields developed in [7] we present a thoughtful study of the theory of a general parallelism structure in an arbitrary real n-dimensional differential manifold. The highlights of our presentation are: (i) *intrinsic* versions of Cartan's first and second structure equations for the torsion and curvature extensors which involve the plus and minus Cartan connection operators, (ii) the concept of deformed (symmetric) parallelism structure and the relative parallelism structure which play, in particular, an important role in the understanding of geometrical theories of the gravitational field.

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